Limits of Trig Functions

- The most important limit involving trig. functions is $\lim_{h\to 0} \frac{\sin h}{h}$. Since $\sin 0 = 0$, substituting $h=0$ leads to an indeterminate form $\frac{0}{0}$.

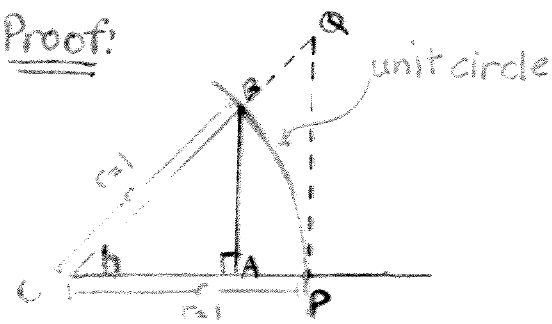
However, the value of $\frac{\sin h}{h}$ as h approaches 0 can be examined in the following table:

	h (radians)	sin h	$\frac{\sin h}{h}$
h approaches 0	0.000 03	0.000 029 999...	0.999 999...
	0.000 02	0.000 019 999...	0.999 999...
	0.000 01	0.000 010 000...	1.000 000...

Therefore, as h gets smaller, the table above

- suggests that as h and sin h get closer and closer to the same value then $\lim_{h\to 0} \frac{\sin h}{h} = 1$

Proof:



From the diagram: (use SOH CAH TOA)

① $\frac{OA}{OB} = \cos h \Rightarrow OA = \cos h$ $(\because OB = 1)$

② $\frac{AB}{OB} = \sin h \Rightarrow AB = \sin h$ $(\because OB = 1)$

③ $\frac{QP}{OP} = \tan h \Rightarrow QP = \tan h$ $(\because OP = 1)$
(in ΔOQP)

Area

Now $|\Delta OAB| < |\text{sector } OPB| < |\Delta OPQ|$

* $\text{Area}_{\Delta} = \frac{1}{2}bh$

$\text{Area}_{\text{sector}} = \frac{1}{2}\theta r^2$

- $\therefore \frac{1}{2}\cos h \sin h < \frac{1}{2}h(1)^2 < \frac{1}{2}(1)\tan h$

* $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\frac{1}{2}\cos h \sin h < \frac{1}{2}h < \frac{1}{2}\frac{\sin h}{\cos h}$

$\therefore \quad \cos h < \frac{h}{\sin h} < \frac{1}{\cos h}$ * divide by $\frac{1}{2}\sin h$

$\frac{1}{\cos h} > \frac{\sin h}{h} > \cos h$ * invert each.

$2 < 3 < 4$

$\frac{1}{2} > \frac{1}{3} > \frac{1}{4}$

$\lim_{h\to 0} \frac{1}{\cos h} \geq \lim_{h\to 0} \frac{\sin h}{h} \geq \lim_{h\to 0} \cos h$ * take the limit of each term

* equal signs appear because areas can be equal with $h \to 0$

$\therefore \quad 1 \geq \lim_{h\to 0} \frac{\sin h}{h} \geq 1$

$\therefore \quad \boxed{\lim_{h\to 0} \frac{\sin h}{h} = 1}$ by the Squeeze Rule

Ex 1. Evaluate the following limits

a) $\lim_{x\to 0} \frac{\sin 3x}{x}$

Notice: $\Rightarrow \frac{0}{0}$ type

$\Rightarrow$ you can not directly use $\lim_{h\to 0} \frac{\sin h}{h} = 1$

$\because$ $3x$ and x are not equal in value

$\therefore$ you need to manipulate

$= \lim_{x\to 0} \frac{3 \sin 3x}{3x}$ * multiply top and bottom by 3

$= 3 \lim_{3x\to 0} \frac{\sin 3x}{3x}$ * note $x \to 0$ then $3x \to 0$

$= 3(1)$ need to be the same.

$= 3$

b) $\lim\limits_{x\to 0} \dfrac{\sin^3 2x}{\sin^3 3x}$ ($\frac{0}{0}$ type)

$= \lim\limits_{x\to 0} \left(\dfrac{\sin 2x}{\sin 3x}\right)^3$ (move exponent to the outside)

$= \lim\limits_{x\to 0} \left(\dfrac{\dfrac{2\cancel{x}(\sin 2x)}{2x}}{\dfrac{3\cancel{x}(\sin 3x)}{3x}}\right)^3$ (* get each sine with a denominator the same as its argument)

* cancel out "x" from $\dfrac{2\cancel{x}}{3\cancel{x}}$

$= \left(\dfrac{2 \lim\limits_{2x\to 0} \dfrac{\sin 2x}{2x}}{3 \lim\limits_{3x\to 0} \dfrac{\sin 3x}{3x}}\right)^3$

$= \left(\dfrac{2}{3}\right)^3$

$= \dfrac{8}{27}$

c) $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x}$ ($\frac{0}{0}$ type)

$= \lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} \cdot \dfrac{\cos x + 1}{\cos x + 1}$

* need this in terms of sin to use $\lim\limits_{h\to 0} \dfrac{\sin h}{h} = 1$

$= \lim\limits_{x\to 0} \dfrac{\cos^2 x - 1}{x(\cos x + 1)}$

* $\sin^2\theta + \cos^2\theta = 1$
$\cos^2\theta - 1 = -\sin^2\theta$

$= \lim\limits_{x\to 0} \dfrac{-\sin^2 x}{x(\cos x + 1)}$

$= -\lim\limits_{x\to 0} \dfrac{\sin x}{x} \cdot \lim\limits_{x\to 0} \dfrac{\sin x}{\cos x + 1}$

$= -(1) \cdot \dfrac{0}{1+1}$

$= 0$

Hwk - handout