Scalar Equation of a Line in $\mathbb{R}^2$

Recall: The vector equation of a line has the form $\vec{r} = (x_0, y_0) + t(a,b)$, where (x_0, y_0) is the position vector for a point on the line and (a,b) is the direction vector of the line. The ***direction vector*** ($\vec{m}$) is any vector that is parallel to the line

We can also determine the equation of a line using a vector that is perpendicular to the line. Such a vector is called a ***normal vector*** ($\vec{n}$).

Since $\vec{m}$ and $\vec{n}$ are perpendicular, then $\vec{m} \cdot \vec{n} = 0$

Eg. 1 Given a line with equation $y = \frac{3}{2}x - 1$, state the direction vector and the vector equation. Also determine a normal vector to the line.

Since the slope of the line is $m = \frac{3}{2}$, then $\vec{m} = (2,3)$

A pt. on the line is y-int $(0,-1)$

$\therefore$ The vector equation of the line is $\vec{r} = (0,-1) + t(2,3)$

Let $\vec{n} = (n_1, n_2)$ be the normal vector.

Then $\vec{m} \cdot \vec{n} = (2,3) \cdot (n_1, n_2) = 0$

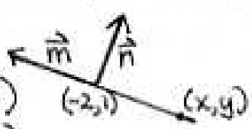
$2n_1 + 3n_2 = 0$

One possible solution is $\vec{n} = (3,-2)$
Another possible solution: $\vec{n} = (-3,2)$
} note: any multiple of $\vec{n}$ is also a normal vector

Notice that, given $\boxed{\vec{m} = (a,b), \text{ then } \vec{n} = (b,-a) \text{ or } \vec{n} = (-b,a)}$

Eg. 2 Determine the equation of the line with normal vector (1,-3) which passes through the point (-2,1). Write the equation of the line in the form Ax + By + C = 0.

Method 1:



Since $\vec{n} = (1,-3)$ then $\vec{m} = (3,1)$

Let (x,y) be a pt. on the line.

$(x+2, y-1) \cdot (1,-3) = 0$

$1(x+2) - 3(y-1) = 0$

$x + 2 - 3y + 3 = 0$

$\boxed{x - 3y + 5 = 0}$

Method 2:

Since $\vec{m} = (3,1)$ and $(-2,1)$ is a point on the line, then the vector equation for the line is: $\vec{r} = (x,y) = (-2,1) + t(3,1)$

Parametric Equations:

$x = -2 + 3t \qquad y = 1 + t$

Symmetric Equation:

$\frac{x+2}{3} = y - 1$

Cross-mult:

$x + 2 = 3(y-1)$

$x + 2 = 3y - 3$

$\boxed{x - 3y + 5 = 0}$

Notice that the coefficients of the x and y terms of the scalar equation are the components of the normal vector. This leads to a 3rd method of determining the scalar equation:

Since $\vec{n}=(1,-3)$ then the scalar equation is of the form $x-3y+C=0$

Since $(-2,1)$ is a point on the line, then

$$(-2)-3(1)+C=0$$
$$-5+C=0$$
$$C=5$$

$$\boxed{\therefore x-3y+5=0}$$

In general, the scalar equation of a straight line in $\mathbb{R}^2$ is written as $Ax+By+C=0$, where the vector $\vec{n}=(A,B)$ is normal to the line and $C=-Ax_0-By_0$ for a point (x_0,y_0) on the line.

Note: There is only one unique normal vector for a line in $\mathbb{R}^2$, but there are infinite normal vectors for a line in $\mathbb{R}^3$. $\therefore$ There is no scalar equation for a line in $\mathbb{R}^3$.

Eg. 3 Determine the scalar equation of the line that passes through the point (-6,4) and is:

a) parallel to $x+4y-8=0$

b) perpendicular to $2x-3y+6=0$

a) From $x+4y-8=0$, $\vec{n}=(1,4)$

The scalar equation is of the form $x+4y+C=0$

Sub pt $(-6,4)=(x,y)$: $(-6)+4(4)+C=0$

$$-6+16+C=0$$
$$C=-10$$

$\therefore x+4y-10=0$ is the scalar equation of the line through $(-6,4)$ and parallel to $x+4y-8=0$

b) For given line, $2x-3y+6=0$, $\vec{n}=(2,-3)$

Since the line we are looking for is perpendicular to this one, then use $\vec{m}=(2,-3)$

Vector equation: $\vec{r}=(x,y)=(-6,4)+t(2,-3)$

Parametric equations: $x=-6+2t \quad y=4-3t$

Symmetric equation: $\dfrac{x+6}{2}=\dfrac{y-4}{-3}$

Cross-multiply: $-3x-18=2y-8$

$$\therefore 3x+2y+10=0$$