Rates of Change

Mathematics deals with relationships between interdependent quantities. We describe these relationships using functions, $y = f(x)$. The dependent variable, y, is a function of the independent variable, x.

Calculus is concerned with how rapidly the *dependent* variable changes when there is a change in the *independent* variable. This is called the rate of change.

Eg. Given that $A = \pi r^2$, how fast does the *area* of a circle change as the *radius* increases?

EG 1 Sean drives from Waterloo to Parry Sound, a distance of 300 km. If the trip takes 4 hours, what is his <u>*average velocity*</u>? → over an interval of time

$$V_{avg} = \frac{\text{distance travelled}}{\text{time}}$$

$$= \frac{300}{4}$$

$$= 75 \text{ km/h}$$

It should be noted that this does <u>not</u> mean that the car's speed is 75km/h at every instant. So, how do we determine the velocity at a **particular instant**?

i) car ⇒ speedometer *ii)* other objects ⇒ not so easy

EG 2 A ball is dropped from a height of 20 m and it takes 2 seconds to hit the ground. What is the velocity ***when it hits the ground***?

In this case we want to know the instantaneous velocity at $t=2$ seconds.

We can approximate the **instantaneous velocity** by examining a *small* time interval (such as $t = 2$ to $t = 2.1$). If $s(t)$ is a function that represents the position of the ball at time t, then

$s(t) = -5t^2$
from physics

$$V_{avg} = \frac{\text{change in position}}{\text{change in time}}$$

$$= \frac{s(2.1) - s(2)}{2.1 - 2}$$

$$= \frac{-5(2.1)^2 + 5(2)^2}{2.1 - 2}$$

$$= \frac{-22.05 + 20}{0.1}$$

$$= -20.5 \text{ m/s}$$

If we determine a time interval as small as possible, we will have the instantaneous velocity:
(i.e. a time interval from $t = 2$ to $t = 2 + h$ and let $h \to 0$)

$$\text{inst. velocity} = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h}$$

$$= \lim_{h \to 0} \frac{(-20 - 20h - 5h^2) - (-20)}{h}$$

$$= \lim_{h \to 0} \frac{-20h - 5h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-20 - 5h)}{\cancel{h}}$$

$$= -20 \text{ m/s}$$

$s(t) = -5t^2$
$s(2) = -5(2)^2$
$= -20$
$s(2+h) = -5(2+h)^2$
$= -5(4 + 4h + h^2)$
$= -20 - 20h - 5h^2$

The example above illustrates that given a function $y = f(x)$, the ***(instantaneous) rate of change*** of y with respect to x when $x = a$ is:

instantaneous: $V_{inst} = \lim_{h \to 0} \dfrac{s(a+h) - s(a)}{h}$

*same as finding the slope of a tangent line

average:

$$V_{avg} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{\text{change in position}}{\text{change in time}}$$

where $t_2 > t_1$

*same as slope of secant line

EG 3 Given $s(t) = t^2 - 2t$ where s is the position of an object in metres and t is the time in seconds:

a) find the average velocity in the third second (ie. from $t = 2$ to $t = 3$)

$$V_{avg} = \frac{s(3) - s(2)}{3-2}$$

$$= \frac{[3^2 - 2(3)] - [2^2 - 2(2)]}{1}$$

$$= \frac{3-0}{1} = 3 \text{ m/s}$$

b) find the instantaneous velocity at $t = 3$

$$V = \lim_{h \to 0} \frac{s(3+h) - s(3)}{h}$$

$$s(3) = 3$$

$$s(3+h) = (3+h)^2 - 2(3+h)$$
$$= 9 + 6h + h^2 - 6 - 2h$$
$$= 3 + 4h + h^2$$

$$= \lim_{h \to 0} \frac{(\cancel{3} + 4h + h^2) - (\cancel{3})}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$$

$$= 4 \text{ m/s}$$