DATE: ____________________

MCB 4UI – *Properties of Limits*

To evaluate the limit of a function for any x-value, we can use the following properties of limits:

Properties of Limits

Given functions $f(x)$ and $g(x)$ and $a \in \mathbb{R}$:

1. $\lim_{x\to a} k =$ K

2. $\lim_{x\to a} x =$ a

3. $\lim_{x\to a} k\cdot f(x) = k\left[\lim_{x\to a} f(x)\right]$

4. $\lim_{x\to a}\left[f(x)\pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to} g(x)$

5. $\lim_{x\to a}\left[f(x)\cdot g(x)\right] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$

6. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

7. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$

E.g. ❶ Evaluate the following limit: (using limit properties)

$$\lim_{x\to 3}\left(2x^2+5x-1\right) = \lim_{x\to 3} 2x^2 + \lim_{x\to 3} 5x - \lim_{x\to 3} 1$$

$$= 2\lim_{x\to 3} x^2 + 5\lim_{x\to 3} x - \lim_{x\to 3} 1$$

$$= 2\left[\lim_{x\to 3} x\right]^2 + 5\lim_{x\to 3} x - \lim_{x\to 3} 1$$

$$= 2(3)^2 + 5(3) - 3 \qquad = 32$$

same

Notice, if we use direct substitution, we get:

$$\lim_{x\to 3}\left(2x^2+5x-1\right) = 2(3)^2 + 5(x) - 1$$

$$= 32$$

In other words, if direct substitution does ***not*** *yield a zero on the denominator*, then it is the case that:

$\lim_{x\to a} f(x) = f(a)$ ← the function values near $x=a$ are the same as the function values at $x = a$

Therefore, we ***can*** use direct substitution to evaluate the limit of a function, if it does not yield a zero on the denominator!!!!

MCV 4U1 Feb. 9, 2012

Evaluating Indeterminate Forms of Limits

Recall: We can evaluate a limit $\lim_{x \to a} f(x)$ by evaluating $f(a)$ if $f(x)$ is continuous at the point $x=a$.

However, sometimes substitution of $x=a$ into $f(x)$ leads to the indeterminate form $\left(\frac{0}{0}\right)$ which is meaningless. We must find a way to rewrite the function so we can evaluate the limit.

Eg1 Evaluate: $\lim_{x \to 3} \frac{x^3 - 27}{x-3}$ $\left[\begin{array}{c}\text{note: sub of } x=3 \\ \text{yields } \frac{0}{0}\end{array}\right]$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{x-3}}$$

*Factor the numerator
$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$$= \lim_{x \to 3} x^2 + 3x + 9$$

$$= (3)^2 + 3(3) + 9$$

*drop limit notation when you substitute

$$= 27$$

Eg2 Evaluate: $\lim_{x \to 9} \frac{x-9}{\sqrt{x} - 3}$ $\left[\frac{0}{0}\right]$

$$= \lim_{x \to 9} \frac{x-9}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3}$$

*Rationalize the denominator

$$= \lim_{x \to 9} \frac{\cancel{(x-9)}(\sqrt{x}+3)}{\cancel{x-9}}$$

$$= \lim_{x \to 9} \sqrt{x} + 3$$

$$= \sqrt{9} + 3$$

$$= 6$$

Eg 3 $\lim\limits_{x \to 5} \dfrac{x^{-1} - 5^{-1}}{x-5}$

$= \lim\limits_{x \to 5} \dfrac{\frac{1}{x} - \frac{1}{5}}{x-5}$ * Get common denominator

$= \lim\limits_{x \to 5} \dfrac{\frac{5-x}{5x}}{x-5}$

Note: $5-x$
$= -1(-5+x)$
$= -1(x-5)$

$= \lim\limits_{x \to 5} \dfrac{5-x}{5x} \cdot \dfrac{1}{x-5}$

$= \lim\limits_{x \to 5} \dfrac{-1\cancel{(x-5)}}{5x} \cdot \dfrac{1}{\cancel{x-5}}$

$= \lim\limits_{x \to 5} \dfrac{-1}{5x}$

$= \dfrac{-1}{5(5)} = \dfrac{-1}{25}$

Eg 4 $\lim\limits_{x \to 0} \dfrac{(x+16)^{1/4} - 2}{x}$

Let $u = (x+16)^{1/4}$
$u^4 = x+16$
$u^4 - 16 = x$

If $x \to 0$
$u^4 - 16 \to 0$
$u^4 \to 16$
$u \to 2$

$= \lim\limits_{u \to 2} \dfrac{u-2}{u^4-16}$

$= \lim\limits_{u \to 2} \dfrac{u-2}{(u^2-4)(u^2+4)}$

$= \lim\limits_{u \to 2} \dfrac{\cancel{u-2}}{\cancel{(u-2)}(u+2)(u^2+4)}$

$= \lim\limits_{u \to 2} \dfrac{1}{(u+2)(u^2+4)}$

$= \dfrac{1}{(2+2)(2^2+4)}$

$= \dfrac{1}{(4)(8)} = \dfrac{1}{32}$