MHF 4U1 - Unit 3 Practice Test Solutions

1. a) $y = \dfrac{x^2(x-3)}{(x-1)^3}$ Restrictions: $\{x \in \mathbb{R} \mid x \neq 1\}$

x-int: $0 = \dfrac{x^2(x-3)}{(x-1)^3}$

$0 = x^2(x-3)$

$x = 0 \quad x = 3$

$\therefore$ x-int at $(0,0)$ & $(3,0)$

y-int: $y = \dfrac{0^2(0-3)}{(0-1)^3}$

$y = 0$

$\therefore$ y-int $(0,0)$

V. Asymptotes: $\lim\limits_{x \to 1^-} \dfrac{x^2(x-3)}{(x-1)^3} \approx \dfrac{(1)(-2)}{-sm} = +\infty$ $\lim\limits_{x \to 1^+} \dfrac{x^2(x-3)}{(x-1)^3} \approx \dfrac{(1)(-2)}{+sm} = -\infty$

$\therefore$ V.A. at $x = 1$

H. Asymptotes: $\lim\limits_{x \to \infty} \dfrac{x^2(x-3)}{(x-1)^3}$

$= \lim\limits_{x \to \infty} \dfrac{x^3 - 3x^2}{x^3 - 3x^2 + 3x - 1}$

$= \lim\limits_{x \to \infty} \dfrac{\cancel{x^3}\left(1 - \frac{3}{x}\right)}{\cancel{x^3}\left(1 - \frac{3}{x} + \frac{3}{x^2} - \frac{1}{x^3}\right)}$

$= 1$

$\lim\limits_{x \to -\infty} \dfrac{x^3 - 3x^2}{x^3 - 3x^2 + 3x - 1}$

$= \lim\limits_{x \to -\infty} \dfrac{x^3\left(1 - \frac{3}{x}\right)}{x^3\left(1 - \frac{3}{x} + \frac{3}{x^2} - \frac{1}{x^3}\right)}$

$= 1$

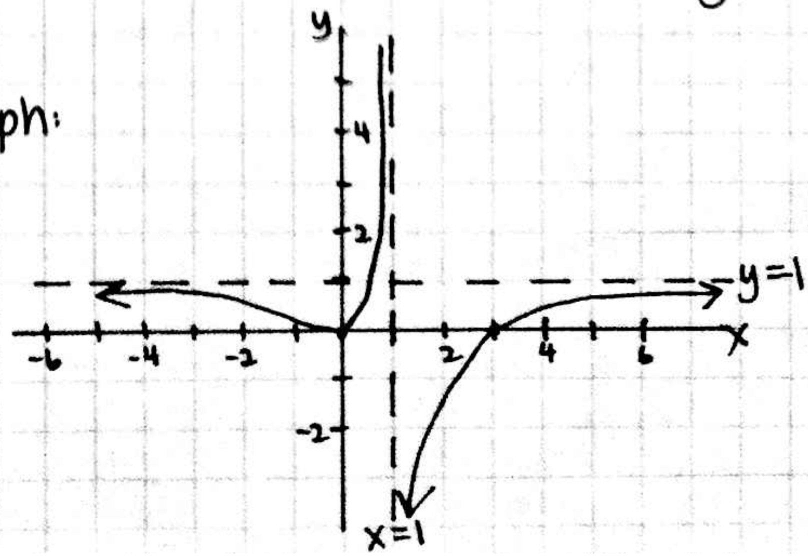
$\therefore$ H.A. at $y = 1$

Test pt $x = 10$:
$y = \dfrac{(10)^2(10-3)}{(10-1)^3}$
$\doteq 0.96 < 1$

Test pt $x = -10$
$y = \dfrac{(-10)^2(-10-3)}{(-10-1)^3}$
$\doteq 0.98 < 1$

There are no oblique asymptotes since the degree of the numerator and denominator are equal.

Graph:



1 b) $y = \dfrac{3x^2-12}{x-1}$

Restrictions: $\{x \in \mathbb{R} \mid x \neq 1\}$

x-int: $0 = \dfrac{3x^2-12}{x-1}$

$0 = 3(x^2-4)$

$0 = 3(x-2)(x+2)$

$x = \pm 2$

$\therefore$ x-int at $(2,0)$ & $(-2,0)$

y-int: $y = \dfrac{3(0)^2-12}{0-1}$

$y = 12$ $\quad \therefore$ y-int $(0,12)$

V. Asymptotes: $\lim\limits_{x\to 1^-} \dfrac{3x^2-12}{x-1}$ $\qquad \lim\limits_{x\to 1^+} \dfrac{3x^2-12}{x-1}$ $\qquad \therefore$ V.A. at $x=1$

$\approx \dfrac{-9}{-sm}$ $\qquad \approx \dfrac{-9}{+sm}$

$\approx +\infty$ $\qquad \approx -\infty$

H.Asymptote $\Rightarrow$ there is no H.A. since the degree of the numerator is one more than the degree of the denominator.

Divide:

$$\begin{array}{c|ccc} 1 & 3 & 0 & -12 \\ & & 3 & 3 \\ \hline & 3 & 3 & -9 \end{array}$$

Oblique Asymptotes:

$\lim\limits_{x\to\infty} \dfrac{3x^2-12}{x-1}$

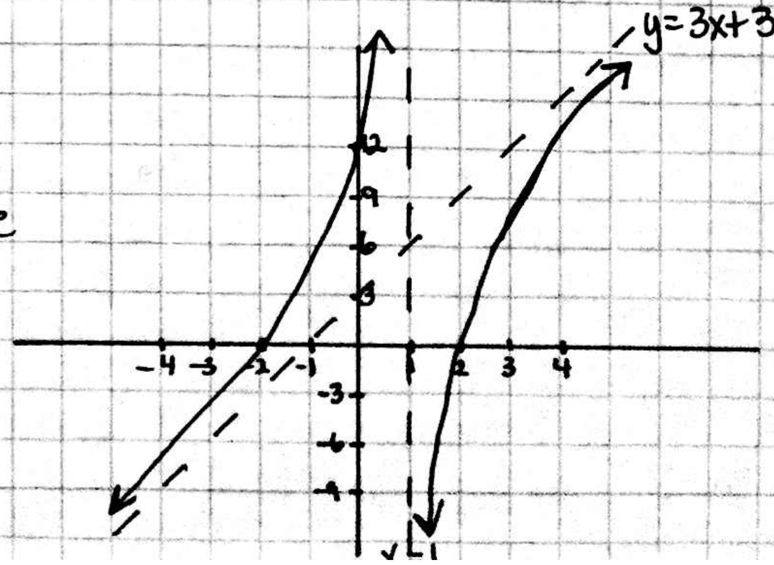
$= \lim\limits_{x\to\infty} 3(x+1) - \underbrace{\dfrac{9}{x-1}}_{\to 0}$

$= +\infty$

$\therefore 3x^2-12 = (x-1)(3x+3) - 9$

$\dfrac{3x^2-12}{x-1} = 3(x+1) - \dfrac{9}{x-1}$

$\therefore$ Equation of O.Asymptote is $y = 3x+3$



2. a) To find pt. of intersection, set $f(x) = g(x)$.

$$\frac{x-2}{x+2} = 4x-1$$

$$x-2 = (x+2)(4x-1)$$
$$x-2 = 4x^2+7x-2$$
$$0 = 4x^2+6x$$
$$0 = 2x(2x+3)$$

$x = 0 \qquad x = \frac{-3}{2}$

Find y-value:

$$g(0) = 4(0)-1 = -1 \qquad g\left(\tfrac{-3}{2}\right) = 4\left(\tfrac{-3}{2}\right)-1 = -7$$

$\therefore$ the curves intersect at $(0,-1)$ and $\left(\frac{-3}{2}, -7\right)$.

b) When $f(x) \geq g(x)$:

$$\frac{x-2}{x+2} \geq 4x-1$$

$$\frac{x-2}{x+2} - 4x-1 \geq 0$$

$$\frac{x-2-(x+2)(4x-1)}{x+2} \geq 0$$

$$\frac{x-2-(4x^2+7x-2)}{x+2} \geq 0$$

$$\frac{-4x^2-6x}{x+2} \geq 0$$

$$h(x) = \frac{-2x(2x+3)}{x+2} \geq 0$$

Find interval boundaries:

Numerator $=0$: $x=0 \quad x = \frac{-3}{2}$

Denominator $=0$: $x=-2$

Interval Chart:

Interval	$-2x$	$2x+3$	$x+2$	$h(x)$
$x \leq -2$	+	−	−	+
$-2 \leq x \leq -\frac{3}{2}$	+	−	+	−
$-\frac{3}{2} \leq x \leq 0$	+	+	+	+
$x \geq 0$	−	+	+	−

$\therefore f(x) \geq g(x)$ when

$$\left\{x \leq -2 \cup -\tfrac{3}{2} \leq x \leq 0\right\}$$

2. c) Analysis for $f(x) = \frac{x-2}{x+2}$ Restrictions: $\{x \in \mathbb{R} \mid x \neq -2\}$

x-int: $0 = \frac{x-2}{x+2}$

$0 = x-2$

$x = 2$

$\therefore$ x-int$(2,0)$

y-int: $y = \frac{0-2}{0+2}$

$= -1$

$\therefore$ y-int$(0,-1)$

V.A. $\lim\limits_{x \to -2^-} \frac{x-2}{x+2}$

$\approx \frac{(-4)}{-sm}$

$\approx +\infty$

$\lim\limits_{x \to -2^+} \frac{x-2}{x+2}$

$\approx \frac{(-4)}{+sm}$

$\approx -\infty$

$\therefore$ V.A. at $x = -2$

H.A. $\lim\limits_{x \to \infty} \frac{x-2}{x+2}$

$= \lim\limits_{x \to \infty} \frac{x(1-2/x)}{x(1+2/x)}$

$= 1$

$\lim\limits_{x \to -\infty} \frac{x-2}{x+2}$

$= \lim\limits_{x \to -\infty} \frac{x(1-2/x)}{x(1+2/x)}$

$= 1$

$\therefore$ H.A at $y = 1$

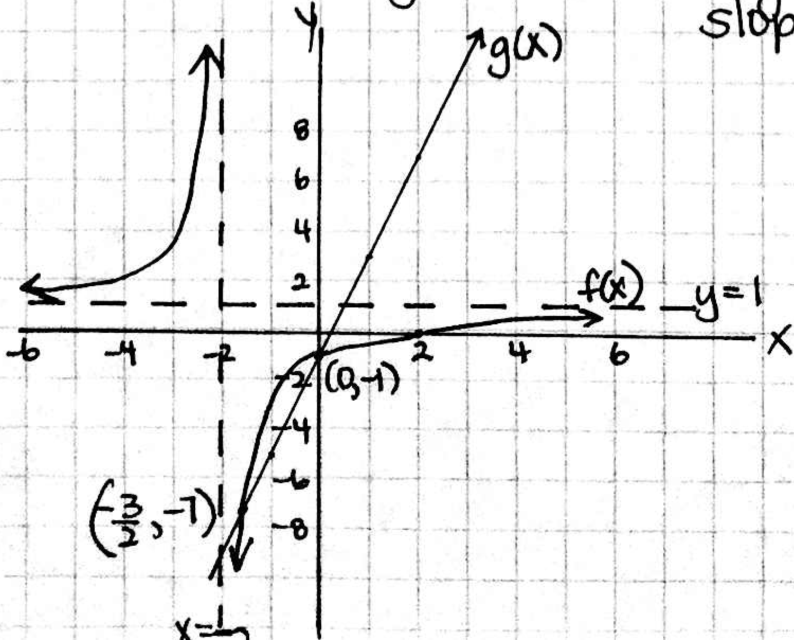
Test pt: $x = 10$
$f(10) \doteq 0.66 < 1$
as $x \to \infty$, $f(x) \to 1$ from below

Test pt $x = -10$
$f(-10) = 1.5 > 1$
as $x \to -\infty$, $f(x) \to 1$ from above

no Oblique asymptotes since degree of numerator equals degree of denominator.

no Analysis needed for $g(x) = 4x - 1$ (graph is linear with slope of 4, y-int of $(0,-1)$).

Graph:



3. $C(t) = \frac{10t}{25+t}$

a) When $C = 3.75$ g/L

$3.75 = \frac{10t}{25+t}$

$3.75(25+t) = 10t$

$93.75 + 3.75t = 10t$

$93.75 = 6.25t$

$t = 15$

$\therefore$ It takes 15 minutes for the salt concentration to reach 3.75 g/L

b) Over a long period of time,

$\lim_{x \to \infty} \frac{10t}{25+t}$

$= \lim_{x \to \infty} \frac{10t}{t(25/t + 1)}$

$= 10$

$\therefore$ Over a long period of time, the salt concentration approaches a limit of 10 g/L.

4. vertical asymptote at $x=5 \Rightarrow (x-5)$ in denominator

x-int at $x = -\frac{1}{2} \Rightarrow$ factor of $(2x+1)$ in numerator

y-int at $x = -\frac{1}{5} \Rightarrow f(0) = -\frac{1}{5}$

horizontal asymptote at $y=2 \Rightarrow \lim_{x \to \infty} f(x) = 2$

Putting this all together gives:

$$f(x) = \frac{2x+1}{x-5}$$

5. $\frac{1-x}{x+5} = \frac{2-x}{x+6}$

$$(1-x)(x+6) = (2-x)(x+5)$$
$$x + 6 - x^2 - 6x = 2x + 10 - x^2 - 5x$$
$$-2x - 4 = 0$$
$$-2x = 4$$
$$x = -2$$