Unit 1: Polynomials 1

Factoring:

Difference of Squares $a^2-b^2=(a-b)(a+b)$

Difference of Cubes $a^3-b^3=(a-b)(a^2+ab+b^2)$

Sum of Cubes $a^3+b^3=(a+b)(a^2-ab+b^2)$

Grouping $ax+ay+bx+by=(a+b)(x+y)$

Long Division:

$$
\begin{array}{r|l}
 & x^2+4x-1 \\
\hline
x+2 & x^3+6x^2+7x-6 \\
 & -(x^3+2x^2) \\
 & \quad 4x^2+7x \\
 & \quad -(4x^2+8x) \\
 & \qquad -x-6 \\
 & \qquad -(-x-2) \\
 & \qquad\quad -4
\end{array}
$$

Synthetic Division:

$$
\begin{array}{r|rrrr}
-2 & 1 & 6 & 7 & -6 \\
 & & -2 & -8 & 2 \\
\hline
 & \underbrace{1 \quad 4 \quad -1}_{\text{Quotient}} & & & \underbrace{-4}_{\text{Remainder}}
\end{array}
$$

$\times$ $+$

*can only use with linear divisors

Division Statement:

$$\underbrace{x^3+6x^2+7x-6}_{\text{Dividend}}=\underbrace{(x+2)}_{\text{Divisor}}\times\underbrace{(x^2+4x-1)}_{\text{Quotient}}\underbrace{-4}_{\text{Remainder}}$$

Dividend = Divisor x Quotient + Remaind

Remainder Theorem:

If a polynomial $f(x)$ is divided by $(x-b)$, remainder is $f(b)$

If a polynomial $f(x)$ is divided by $(ax-b)$, remainder is $f\left(\frac{b}{a}\right)$

Factor Theorem:

A polynomial $f(x)$ has a factor of $(x-b)$ if and only if $f(b)=0$

If $(x-b)$ is a factor of $f(x)$ then b is a factor of the constant term of $f(x)$.

Eg. Factor $x^4-2x^2+16x-15$

Possibilities for b: $\pm1, \pm3, \pm5$

$f(1)=(1)^4-2(1)^2+16(1)-15=0$

$\therefore (x-1)$ is a factor

change sign

Divide:

note 0 for missing x^3 term

$$
\begin{array}{c|ccccc}
1 & 1 & 0 & -2 & 16 & -15 \\
 & & 1 & 1 & -1 & 15 \\
\hline
 & 1 & 1 & -1 & 15 & 0
\end{array}
$$

$f(x)=(x-1)\underbrace{(x^3+x^2-x+15)}$

call this g(x) - use factor theorem again

$g(-3) = (-3)^3 + (-3)^2 - (-3) + 15$
$= 0$

$\therefore (x+3)$ is a factor

Divide:

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -1 & 15 \\ & & -3 & 6 & -15 \\ \hline & 1 & -2 & 5 & 0 \end{array}$$

$\underbrace{1 \quad -2 \quad 5}_{x^2-2x+5}$

$\therefore f(x) = x^4 - 2x^2 + 16x - 15$
$= (x-1)(x+3)(x^2-2x+5)$

Solving Polynomial Equations

*Use Factor Theorem and Quadratic Formula to solve equations of degree higher than 2 for $x \in \mathbb{C}$ (complex numbers)

Eg. Solve $8x^3 - 27 = 0$

$(2x-3)(4x^2+6x+9) = 0$

$2x - 3 = 0$
$x = \frac{3}{2}$

$\underbrace{4x^2+6x+9}_{\text{Solve with Quadratic Formula}} = 0$

$x = \frac{-6 \pm \sqrt{6^2 - 4(4)(9)}}{2(4)}$

$= \frac{-6 \pm \sqrt{-108}}{4}$

$= \frac{-6 \pm 6i\sqrt{3}}{4} = \frac{-3 \pm 3i\sqrt{3}}{2}$

Eg. Determine the polynomial $f(x)$ with complex roots $x = \pm 2i$, x-int at $x = 2$ and $x = -3/2$ and given $f(-2) = 32$.

Let $f(x) = k(x-2i)(x+2i)(x-2)(2x+3)$
$= k(x^2+4)(x-2)(2x+3)$
$f(-2) = k((-2)^2+4)(-2-2)(2(-2)+3) = 32$
$k = 1 \Rightarrow f(x) = (x^2+4)(x-2)(2x+3)$

Practice Questions

1. Factor:

a) $8x^3+125$ b) $4x^2-4x-49y^2+1$ c) $2x^3-x^2-22x-24$

2. Divide:

a) $(x^4+5x^3-4x^2-19x+5)\div(x+5)$

b) $(x^4+3x^3-2x^2+5x-1)\div(x^2+7)$

3. Determine the value of k if $f(x)=12x^3+kx^2-x-6$ has a factor of $(2x-1)$

4. When $2x^3-mx^2+nx-2$ is divided by $x+1$ the remainder is -12. Also $x-2$ is a factor of the function. Determine values for m and n.

5. Solve for $x\in\mathbb{C}$:

a) $x^4-6x^3+2x^2-12x=0$

b) $4x^4+6x^3-6x^2-4x=0$

Unit 1: Polynomials 1 Practice Questions

1. a) $8x^3+125=(2x+5)(4x^2-10x+25)$

b) $4x^2-4x+1-49y^2=(2x-1)^2-49y^2$
$=(2x-1-7y)(2x-1+7y)$

c) $2x^3-x^2-22x-24=f(x)$

$f(-2)=2(-2)^3-(-2)^2-22(-2)-24=0$
$\therefore$ $x+2$ is a factor

$$\begin{array}{r|rrrr} -2 & 2 & -1 & -22 & -24 \\ & & -4 & 10 & 24 \\ \hline & 2 & -5 & -12 & 0 \end{array}$$

$\therefore f(x)=(x+2)(2x^2-5x-12)$
$=(x+2)(2x+3)(x-4)$

2. a)

$$\begin{array}{r|rrrrr} -5 & 1 & 5 & -4 & -19 & 5 \\ & & -5 & 0 & 20 & -5 \\ \hline & 1 & 0 & -4 & 1 & 0 \end{array}$$

$\therefore x^4+5x^3-4x^2-19x+5=(x+5)(x^3-4x+1)$

b)

$$\begin{array}{rl} & \phantom{x^4+3x^3-{}}x^2+3x-9 \\ x^2+7 & \overline{)\,x^4+3x^3-2x^2+5x-1} \\ & \underline{x^4+7x^2} \\ & \phantom{x^4+{}}3x^3-9x^2+5x \\ & \phantom{x^4+{}}\underline{3x^3+21x} \\ & -9x^2-16x-1 \\ & \underline{-9x^2-63} \\ & -16x+62 \end{array}$$

$\therefore x^4+3x^3-2x^2+5x-1=(x^2+7)(x^2+3x-9)+(-16x+62)$

3. If $(2x-1)$ is a factor of $f(x)$ then $f(\frac{1}{2})=0$

$$f(\tfrac{1}{2})=12(\tfrac{1}{2})^3+k(\tfrac{1}{2})^2-(\tfrac{1}{2})-6=0$$
$$12(\tfrac{1}{8})+k(\tfrac{1}{4})-(\tfrac{1}{2})-6=0$$
$$(\tfrac{1}{4})k=6+\tfrac{1}{2}-\tfrac{3}{2}$$
$$(\tfrac{1}{4})k=5$$
$$k=20$$

4. $f(x)=2x^3-mx^2+nx-2$

Given $(x-2)$ is a factor, then $f(2)=0$

$$f(2)=2(2)^3-m(2)^2+n(2)-2=0$$
$$16-4m+2n-2=0$$
$$14=4m-2n$$
$$\div 2 \quad 7=2m-n \quad (1)$$

Given the remainder is -12 when dividing by $x+1$ then $f(-1)=-12$

$$f(-1)=2(-1)^3-m(-1)^2+n(-1)-2=0$$
$$-2-m-n-2=0$$
$$-4=m+n \quad (2)$$

Add:
$$\begin{aligned} 7&=2m-n \quad (1)\\ -4&=m+n \quad (2)\\ \hline 3&=3m\\ m&=1 \end{aligned}$$

Sub $m=1$ in (2)
$$-4=(1)+n$$
$$-5=n$$

$\therefore m=1$ & $n=-5$

5. a) $x^4 - 6x^3 + 2x^2 - 12x = 0$

$x(\underbrace{x^3 - 6x^2 + 2x - 12}_{\text{factor by grouping or factor theorem}}) = 0$

$x\left[x^2(x-6) + 2(x-6)\right] = 0$

$x(x-6)(x^2+2) = 0$

$x = 0 \quad x = 6 \quad x^2 = -2$

$x = \pm i\sqrt{2}$

b) $4x^4 + 6x^3 - 6x^2 - 4x = 0$

$2x(\underbrace{2x^3 + 3x^2 - 3x - 2}_{g(x)}) = 0$

Find factors of $g(x)$: $g(1) = 2(1)^3 + 3(1)^2 - 3(1) - 2 = 0$

$\therefore$ $x - 1$ is a factor

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -3 & -2 \\ & & 2 & 5 & 2 \\ \hline & 2 & 5 & 2 & 0 \end{array}$$

$\therefore$ $4x^4 + 6x^3 - 6x^2 - 4x = 0$

$2x(x-1)(2x^2 + 5x + 2) = 0$

$2x(x-1)(2x+1)(x+2) = 0$

$x = 0 \quad x = 1 \quad x = -\frac{1}{2} \quad x = -2$