MCV 4U1

Implicit Differentiation

So far we have differentiated functions where y is defined explicitly in terms of x.

eg. $y = 5x^2 - 3x + 1$ or $y = \sqrt{2x-1}$

However, some functions are defined implicitly as a relation between x and y.

eg. $x^2 + y^2 = 25$

Consider the problem of differentiating the above relation: $x^2 + y^2 = 25$

We could rewrite the function to define y explicitly in terms of x:

$$y^2 = 25 - x^2$$

$$y = \pm\sqrt{25 - x^2}$$

Differentiating both sides w.r.t. x gives:

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\pm\sqrt{25-x^2}\right)$$

this notation means "differentiate with respect to x"

$$\frac{dy}{dx} = \pm\frac{1}{2}(25-x^2)^{-1/2}(-2x)$$

$$\frac{dy}{dx} = \pm\frac{-x}{\sqrt{25-x^2}}$$

Unfortunately, we can not always write a relationship to express y explicitly in terms of x. In these cases, we use implicit differentiation.

Eg1
Back to $x^2+y^2=25$

This time we will differentiate both sides implicitly w.r.t. x:

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(25)$$

$$2x+\overbrace{2y\cdot\frac{dy}{dx}}=0$$

Note:
$y=f(x)$
$y^2=[f(x)]^2$
Using chain rule,
$\frac{d}{dx}(y^2)=2f(x)\cdot f'(x)$
$=2y\cdot\frac{dy}{dx}$

Rearrange to solve for $\frac{dy}{dx}$:

$$2y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=\frac{-2x}{2y}$$

$$\frac{dy}{dx}=\frac{-x}{y}$$

Compare to previous answer: $\frac{dy}{dx}=\pm\frac{-x}{\sqrt{25-x^2}}$
Since $y=\pm\sqrt{25-x^2}$
then $\frac{dy}{dx}=\frac{-x}{y}=\frac{-x}{\pm\sqrt{25-x^2}}$

Eg2 Find $\frac{dy}{dx}$ for:

$$2xy-y^3=4$$

$$\frac{d}{dx}(2xy)-\frac{d}{dx}(y^3)=\frac{d}{dx}(4)$$

(product rule)

$$\left[(2)y+2x\frac{dy}{dx}\right]-\left[3y^2\frac{dy}{dx}\right]=0$$

$$2x\frac{dy}{dx}-3y^2\frac{dy}{dx}=-2y$$

$$\frac{dy}{dx}=\frac{-2y}{2x-3y^2}$$

Eg3 Find the slope of the tangent to the ellipse $x^2+4y^2=25$ at the pt. $(3,-2)$

Differentiate both sides w.r.t. x:

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(4y^2)=\frac{d}{dx}(25)$$

$$2x + 8y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}=\frac{-2x}{8y}$$

$$\frac{dy}{dx}=\frac{-x}{4y}$$

At the pt. $(3,-2)$: $\left.\frac{dy}{dx}\right|_{\substack{x=3\\y=-2}} = \frac{-(3)}{4(-2)} = \frac{3}{8}$

$\therefore$ The slope of the tangent to the ellipse at the pt. $(3,-2)$ is $\frac{3}{8}$.

HW: pg 178 #2 odds, 3ab, 4