Equations of a Line in $\mathbb{R}^3$

Recall: Previously, we determined the following representations of a line in $\mathbb{R}^2$:

given position vector (x_0, y_0) and direction vector (m_1, m_2)

Vector Equation: $\vec{r} = (x, y) = (x_0, y_0) + t(m_1, m_2)$

Parametric Equations: $x = x_0 + tm_1 \quad y = y_0 + tm_2$

Symmetric Equation: $\dfrac{x - x_0}{m_1} = \dfrac{y - y_0}{m_2}$

Scalar Equation: $Ax + By + C = 0$

We can easily extend our knowledge of lines to $\mathbb{R}^3$. Note: Since there is **not** a unique normal to a line in $\mathbb{R}^3$, then lines in $\mathbb{R}^3$ cannot be expressed in scalar form.

Eg. 1 Given the points A(2, -2, -8) and B(5, -2, -14), determine the vector equation, parametric equations and symmetric equation of the line that passes through A and B.

First find the direction vector $\vec{m} = \vec{AB} = (5, -2, -14) - (2, -2, -8)$
$= (3, 0, -6)$
$= 3(1, 0, -2)$
$\therefore \vec{m} = (1, 0, -2)$

Vector equation: $\vec{r} = (2, -2, -8) + t(1, 0, -2)$

Parametric equations: $x = 2 + t \quad y = -2 \quad z = -8 - 2t$

Symmetric equation:

similar to: $\frac{y+2}{0}$ (und.)

$$\frac{x-2}{1} = \frac{z+8}{-2} \; ; \; y = -2$$

state separately since can't solve for t in $y = -2$

Eg. 2 Determine the vector equation of a line given its symmetric equation is $-x = y + 2 = z$

Rewrite symmetric equation:

top values give pt. on line

$$\frac{x-0}{-1} = \frac{y+2}{1} = \frac{z-0}{1}$$

bottom values give $\vec{m}$

$\therefore \vec{m} = (-1, 1, 1)$

pt on line: $(0, -2, 0)$

$\therefore$ vector equation: $\vec{r} = (0, -2, 0) + t(-1, 1, 1)$

In general, the equation of a line with position vector $\vec{r_0} = (x_0, y_0, z_0)$ and direction vector $\vec{m} = (m_1, m_2, m_3)$ can be represented as:

1) $\vec{r} = (x_0, y_0, z_0) + t(m_1, m_2, m_3)$ * Vector Equation

2) $\left.\begin{array}{l} x = x_0 + tm_1 \\ y = y_0 + tm_2 \\ z = z_0 + tm_3 \end{array}\right]$ * Parametric Equations

3) $\dfrac{x - x_0}{m_1} = \dfrac{y - y_0}{m_2} = \dfrac{z - z_0}{m_3}$ * Symmetric Equation

Eg. 3 Determine if the following lines are the same (coincident), parallel, or neither:

$\ell_1 : (x, y, z) = (1, 4, -3) + k(6, -2, 4)$

$\ell_2 : (x, y, z) = (-5, 6, 7) + t(-3, 1, -2)$

From l_1, $\vec{m}_1 = (6, -2, 4)$ From l_2, $\vec{m}_2 = (-3, 1, -2)$

$= -2(-3, 1, -2)$

$\because \vec{m}_1 = -2\vec{m}_2$ then either $\underbrace{l_1 = l_2}_{\text{coincident}}$ or $\underbrace{l_1 \parallel l_2}_{\text{parallel}}$

To determine if $l_1 = l_2$ then we need to check if a point from one line is on the other.

A point on l_2 is $(-5, 6, 7)$. If this point is on l_1 then $(-5, 6, 7) = (1, 4, -3) + k(6, -2, 4)$

Set corresponding components equal:

$-5 = 1 + 6k$ $\quad 6 = 4 - 2k$ $\quad 7 = -3 + 4k$

$-6 = 6k$ $\quad 2 = -2k$ $\quad 10 = 4k$

$k = -1$ $\quad k = -1$ $\quad k = \frac{10}{4} = \frac{5}{2}$

$\because$ The values for k are not all the same, then $(-5, 6, 7)$ is not on l_1 so $l_1 \neq l_2$.

$\therefore l_1$ is parallel to l_2.