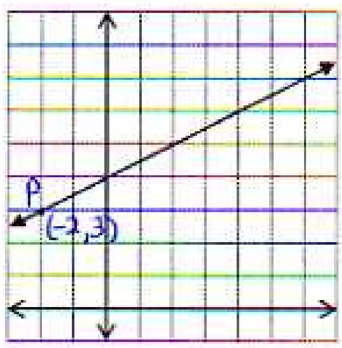
Parametric, Vector & Symmetric Equations of a Line in $\mathbb{R}^2$

Background review – a comparison of lines and vectors:

Lines	*Vectors*
-defined in two directions	-defined in one direction only
-infinite length (but a line segment has finite length)	-finite magnitude
-location is fixed	-no fixed location
-two lines are coincident if they have same direction and location	-two vectors are equal if they have the same direction & magnitude

Note: Lines are **not** vectors, but vectors can be used to describe a line.

Recall: To find the equation of a line in 2-space, you need to know a point on the line and the slope of the line.



From the graph, a pt. on the line is $P(-2,3)$ and $m=\frac{1}{2}$.

Equation: $\frac{y-3}{x+2}=\frac{1}{2}$

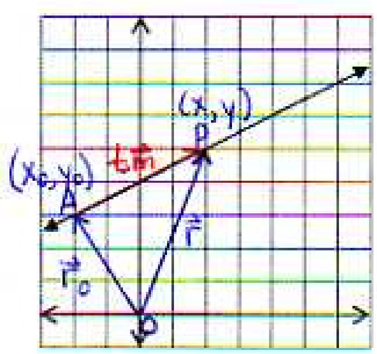
$2y-6=x+2$

$x-2y+8=0$ (standard form)

$-2y=-x-8$

$y=\frac{1}{2}x+4$ (slope y-int form)

The same line can be drawn using vectors as follows:



Let $A(x_0,y_0)$ be a point on the line.

The position vector $\overrightarrow{OA}$ defines a *particular* point on the line.

The position vector $\overrightarrow{OP}$ defines any *general* point on the line.

The vector $\vec{m}$ is the *direction vector* for the line. A line with direction vector $\vec{m}=(a,b)$ has slope $m=\frac{b}{a}$, provided that $a\neq 0$.

Note: The direction vector, $\vec{m}$, of a line is not unique. Any scalar multiple of $\vec{m}$, ie. $t\vec{m}$, $t\in\mathbb{R}$ is also a direction vector. For example, $\vec{m}=(2,1)$ is equivalent to $(4,2)=2(2,1)$. Always reduce a direction vector to lowest terms.

Vector Equation of a Line

From the previous diagram: $\overrightarrow{OP} = \overrightarrow{OA} + t\vec{m}$

OR $(x, y) = (x_0, y_0) + t(a, b)$ $\because \vec{m} = (a, b)$

OR $\vec{r} = \vec{r}_0 + t\vec{m}$

In our example,

$$\vec{r} = (x, y) = \underbrace{(-2, 3)}_{\text{position vector}} + t\,\underbrace{(2, 1)}_{\text{direction vector}} \Rightarrow \text{vector equation}$$

Eg. 1 State the direction vector for each of the following lines:

a) the line passing through the points A(-3, 4) and B(7, 2)

$\vec{m} = \overrightarrow{AB} = (7, 2) - (-3, 4) = (10, -2)$
$= 2(5, -1)$ $\therefore \vec{m} = (5, -1)$

b) a line with slope $m = -\frac{5}{3}$

$\vec{m} = (3, -5)$ OR $\vec{m} = (-3, 5)$

c) a vertical line through the point (-5, 7)

$\because$ slope of a vertical line is undefined
ie. $m = \frac{1}{0} \Rightarrow \vec{m} = (0, 1)$

Parametric Equations

From the vector equation: $(x, y) = (x_0, y_0) + t(a, b)$

If we equate corresponding components, we obtain the parametric equations of a line:

$(x, y) = (x_0, y_0) + (ta, tb)$
$= (x_0 + ta, y_0 + tb)$

$\therefore \left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + yt \end{aligned}\right\}$ parametric equations

* note t = parameter

Symmetric Equations

From the parametric equations above, isolating each equation for t yields the symmetric equations for a line.

Solve each of the above equations for t:

$\frac{x - x_0}{a} = t$ $\frac{y - y_0}{b} = t$ $\therefore \frac{x - x_0}{a} = \frac{y - y_0}{b}$

* symmetric equation

Eg. 2 A line passes through the points A(5, -2) and B(2, 6). Determine:

a) the vector equation for the line

direction vector: $\vec{m} = \vec{AB} = (2,6)-(5,-2) = (-3,8)$

vector equation: $\vec{r} = (5,-2) + t(-3,8) \quad t \in \mathbb{R}$

use any pt on line

b) the parametric equations for the line

from above $(x,y) = (5,-2) + t(-3,8)$

$\therefore x = 5-3t$

$y = -2+8t$

c) the symmetric equation for the line

$$\frac{x-5}{-3} = \frac{y+2}{8}$$

d) another point on the line

Let $t=5$ (choose any value for t; sub into parametric equations)

$x = 5-3(5)$ $\quad y = -2+8(5)$

$x = -10$ $\quad\quad = 38$

$\therefore$ pt $(-10,38)$ is on the line

e) the equation of the line in the form $y = mx + b$

Since $\vec{m} = (-3,8)$ then $m = -\frac{8}{3}$

To find "b" (y-int) set $x=0$ and solve for t: $\quad 0 = 5-3t$

$3t = 5 \Rightarrow t = \frac{5}{3}$

sub into $y = -2+8t$

$y = -2+8\left(\frac{5}{3}\right)$

$= -2 + \frac{40}{3} = \frac{34}{3}$ "b"

$$\boxed{\therefore y = -\frac{8}{3}x + \frac{34}{3}}$$

Eg. 3 Are the lines represented by the following vector equations coincident?

same line

$\vec{r}_1 = (3,4) + s(2,-1)$

$\vec{r}_2 = (-9,10) + t(-6,3)$

Check direction vectors first:

$\vec{m}_1 = (2,-1)$ $\quad \vec{m}_2 = (-6,3) = -3(2,-1)$

$\because \vec{m}_2 = -3\vec{m}_1$ then the lines are parallel $(l_1 \parallel l_2)$

To check if the lines are coincident, check if the point from one line is on the other.

Pt (3,4) is on the first line. If it is also on the second line, then $(3,4) = (-9,10) + t(-6,3)$

$\therefore 3 = -9-6t$ $\quad 4 = 10+3t$

$12 = -6t$ $\quad -6 = 3t$

$t = -2$ $\quad t = -2$

$\because t = -2$ for both then (3,4) is on the other line

$\therefore$ lines are coincident $(l_1 = l_2)$

Eg. 4 Determine a vector equation for the line that is perpendicular to $\vec{r} = (4,1) + t(-3,2),\ t \in R$ and passes through the point P(6, 5).

From given line, $\vec{m} = (-3,2)$
then the direction vector
for a perpendicular line
is $\perp \vec{m} = (2,3)$

$\therefore$ vector equation is

$$\vec{r} = (6,5) + t(2,3)$$

Recall: Given $\vec{m} = (a,b)$

$\Rightarrow m = \frac{b}{a}$ (in $\mathbb{R}^2$ only)

$\therefore \perp m = \frac{-a}{b}$ or $\frac{a}{-b}$

so $\perp \vec{m} = (-b,a)$ or $(b,-a)$