

## The Chain Rule

Consider the function:  $f(x) = (2x+3)^3$

Find the derivative by first expanding:

$$\begin{aligned}f(x) &= (2x+3)(2x+3)(2x+3) \\ &= (2x+3)(4x^2+12x+9) \\ &= 8x^3+24x^2+18x+12x^2+36x+27 \\ &= 8x^3+36x^2+54x+27 \\ f'(x) &= 24x^2+72x+54\end{aligned}$$

Another way of looking at this:

Consider  $f(x)$  as a composite function:

ie.  $f(x) = g(k(x))$  when  $g(x) = x^3$      $k(x) = 2x+3$   
 $g'(x) = 3x^2$      $k'(x) = 2$

Note:  $g(1) = (1)^3$   
 $g(a) = (a)^3$   
 $g(k(x)) = (2x+3)^3$

Also:  $g'(k(x)) = 3(2x+3)^2$   
 $= 3(4x^2+12x+9)$   
 $= 12x^2+36x+27 \Rightarrow$  compare to  $f'(x)$   
above

From above,  $f'(x) = 24x^2+72x+54$

Since  $g'(k(x)) \cdot k'(x) = 3(2x+3)^2 (2)$   
 $= 3(4x^2+12x+9)(2)$   
 $= 24x^2+36x+27$

$\therefore$  When  $f(x) = g(k(x))$  then  $f'(x) = g'(k(x)) \cdot k'(x)$

CHAIN RULE

Another way to express the chain Rule (using Leibniz notation):

$$\text{If } y = (2x+3)^3$$

$$\text{let } u = 2x+3$$

$$\text{then } y = u^3$$

$$\frac{dy}{du} = 3u^2$$

$$\frac{du}{dx} = 2$$

$$\text{So: } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 3u^2 (2)$$

$$= 6u^2$$

$$\frac{dy}{dx} = 6(2x+3)^2$$

Eg1 Find the derivative.

$$(a) \ y = (5x^2-3)^6$$

$$\frac{dy}{dx} = 6(5x^2-3)^5 (10x)$$

$$= 60x(5x^2-3)^5$$

$$(b) \ y = \sqrt{2x^3-5x} = (2x^3-5x)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(2x^3-5x)^{-1/2} (6x^2-5)$$

$$= \frac{6x^2-5}{2(2x^3-5x)^{1/2}}$$

Eg2 Given  $y = (\sqrt{x} - 3x^2)^4$  determine  $\frac{dy}{dx} \Big|_{x=1}$

$$\frac{dy}{dx} = 4(\sqrt{x} - 3x^2)^3 \left( \frac{1}{2}x^{-1/2} - 6x \right)$$

$$\frac{dy}{dx} \Big|_{x=1} = 4(\sqrt{1} - 3(1)^2)^3 \left( \frac{1}{2}(1)^{-1/2} - 6(1) \right)$$

$$= 4(-2)^3 \left( -\frac{11}{2} \right) = 176$$

Eg 3 Differentiate but do not simplify.

$$(a) y = (3x^2 - 4)^5 (x - 2x^4)^3$$

$$\frac{dy}{dx} = [5(3x^2 - 4)^4 (6x)] (x - 2x^4)^3 + (3x^2 - 4)^5 [3(x - 2x^4)^2 (1 - 8x^3)]$$

$$(b) f(x) = \sqrt{\frac{x^3 - 2}{6x + 5}} = \left[ \frac{x^3 - 2}{6x + 5} \right]^{1/2}$$

$$f'(x) = \frac{1}{2} \left[ \frac{x^3 - 2}{6x + 5} \right]^{-1/2} \left[ \frac{(3x^2)(6x + 5) - (x^3 - 2)(6)}{(6x + 5)^2} \right]$$