Max/Min Word Problems (Additional Review) – Solutions

1. a) Given $f(x)=\dfrac{1+x}{1-x}$, for $2\le x\le 5$

$$f'(x)=\frac{(1-x)-(-1)(1+x)}{(1-x)^2}$$
$$=\frac{2}{(1-x)^2}$$

For critical numbers,

(i) $f'(x)=\dfrac{2}{(1-x)^2}=0$ (ii) $f'(x)$ is undefined when x=1 (not part of domain)

no solution

Check endpoints: $f(2)=\dfrac{1+(2)}{1-(2)}=\dfrac{3}{-1}=-3$ (min. value)

$f(5)=\dfrac{1+(5)}{1-(5)}=\dfrac{6}{-4}=\dfrac{3}{-2}$ (max. value)

$\therefore$ There is an absolute max. pt. at $\left(5,-\frac{3}{2}\right)$ and an absolute min. pt. at $(2,-3)$.

b) Given $g(x)=2\sin x+\cos 2x$, for $-\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2}$

$$g'(x)=2\cos x-2\sin 2x$$
$$=2\cos x-2(2\sin x\cos x)$$
$$=2\cos x(1-2\sin x)$$

For critical numbers,

(i) $g'(x)=2\cos x(1-2\sin x)=0$ (ii) $g'(x)$ is defined for all values of x in the domain

$\cos x=0$ or $1-2\sin x=0$

$x=\dfrac{\pi}{2},-\dfrac{\pi}{2}$ $x=\dfrac{\pi}{6}$

Check values: $g\left(-\frac{\pi}{2}\right)=2\sin\left(-\frac{\pi}{2}\right)+\cos\left[2\left(-\frac{\pi}{2}\right)\right]=-3$ (min. value)

$g\left(\frac{\pi}{6}\right)=2\sin\left(\frac{\pi}{6}\right)+\cos\left[2\left(\frac{\pi}{6}\right)\right]=\dfrac{3}{2}$ (max. value)

$g\left(\frac{\pi}{2}\right)=2\sin\left(\frac{\pi}{2}\right)+\cos\left[2\left(\frac{\pi}{2}\right)\right]=1$

$\therefore$ There is an absolute max. pt. at $\left(\frac{\pi}{6},\frac{3}{2}\right)$ and an absolute min. pt. at $\left(-\frac{\pi}{2},-3\right)$.

2. Let x be the number of music players sold. Let p be the price per player.

Given $(x_1, p_1) = (8000, 50)$ and $(x_2, p_2) = (7900, 51)$

then $m = \frac{50-51}{8000-7900} = \frac{-1}{100}$

For the price function, $\frac{-1}{100} = \frac{p-50}{x-8000}$

$$100p - 5000 = -x + 8000$$

$$100p = -x + 13000$$

$$p = \frac{-1}{100}x + 130$$

Revenue function: $R(x) = x \cdot p(x)$

$$= x\left(\frac{-1}{100}x + 130\right)$$

$$= \frac{-1}{100}x^2 + 130x$$

$$R'(x) = \frac{-1}{50}x + 130$$

For critical numbers,

(i) $R'(x) = \frac{-1}{50}x + 130 = 0$

$$130 = \frac{1}{50}x$$

$$x = 6500$$

(ii) $R'(x)$ is defined for all x values in the domain

Find price: $p(6500) = \frac{-1}{100}(6500) + 130 = \$65$

$\therefore$ They should charge \$65 per player in order to maximize revenue.

3. a) Given $C(x) = 280000 + 12.5x + 0.07x^2$,

The average cost function is given by $c(x) = \dfrac{280000 + 12.5x + 0.07x^2}{x}$.

When 1000 items are produced,

$$c(1000) = \frac{280000}{1000} + 12.5 + 0.07(1000)$$
$$= \$362.50\text{/item}$$

The marginal cost is given by $C'(x) = 12.5 + 0.14x$

When 1000 items are produced, $C'(1000) = 12.5 + 0.14(1000)$

$$= \$152.50\text{/item}$$

b) For minimum average cost, find the derivative: $c'(x) = \dfrac{-280000}{x^2} + 0.07$

Solve $c'(x) = 0$:

$$0.07 = \frac{280000}{x^2}$$
$$0.07x^2 = 280000$$
$$x^2 = 4000000$$
$$x = 2000$$

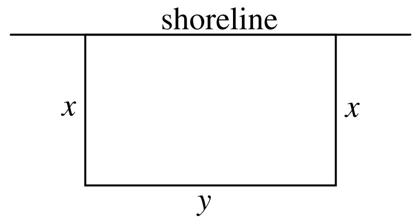
$\therefore$ A production level of 2000 items will minimize average cost.

c) When 2000 items are produced,

$$c(2000) = \frac{280000}{(2000)} + 12.5 + 0.07(2000)$$
$$= \$292.50\text{/item}$$

$\therefore$ The minimum average cost is \$292.50/item when 2000 items are produced.

4. a) Let x be the width of the rectangle and let y be the length.



Max $A = x \cdot y$ given $2x + y = 400$

$y = 400 - 2x$ (sub into $A = x \cdot y$)

$$A = x(400 - 2x)$$
$$= 400x - 2x^2 \quad (0 \le x \le 200)$$
$$A'(x) = 400 - 4x$$

For critical numbers,

(i) $A'(x) = 400 - 4x = 0$
$400 = 4x$
$x = 100$

(ii) $A'(x)$ is defined for all values of x in the domain

Check: $A(0) = 400(0) - 2(0)^2 = 0$

$A(100) = 400(100) - 2(100)^2 = 20000$ (max.)

$A(200) = 400(200) - 2(200)^2 = 0$

Find y: $y = 400 - 2(100) = 200$

$\therefore$ The maximum swimming area is 20000 m^2 when the dimensions are 100 m x 200 m

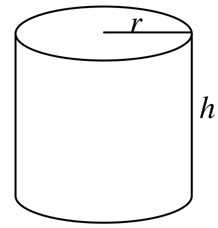
b) Since the critical number from part a) is no longer part of the domain $(0 \le x \le 50)$, check new endpoints:

$A(0) = 0$

$A(50) = 400(50) - 2(50)^2 = 15000$

$\therefore$ The new maximum area is 15000 m^2 when the dimensions are 50 m x 300 m.

5. Let r be the radius and h be the height of the can.



Since the cost (C) of the materials is based on the surface area (A), we need to find the minimum value of

$A = 2\pi r^2 + 2\pi rh$ given $V = \pi r^2 h = 1000$ cm^3

$$h = \frac{1000}{\pi r^2} \quad \text{(sub into } A\text{)}$$

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$A = 2\pi r^2 + \frac{2000}{r} \qquad (r > 0)$$

$$A' = 4\pi r - \frac{2000}{r^2}$$

For critical numbers,

(i) $A' = 4\pi r - \dfrac{2000}{r^2} = 0$

$$4\pi r = \frac{2000}{r^2}$$

$$4\pi r^3 = 2000$$

$$r = \sqrt[3]{\frac{2000}{4\pi}}$$

$$r \approx 5.42$$

(ii) $A'(x)$ is undefined when $r = 0$

(but $r > 0$)

Check using Second Derivative Test: $A'' = 4\pi + \dfrac{4000}{r^3}$

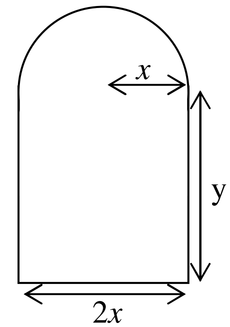
when r = 5.42, $A''(5.42) = 4\pi + \dfrac{4000}{(5.42)^3} > 0$

$\therefore$ a min exists at $r = 5.42$

Find h: $h = \dfrac{1000}{\pi(5.42)^2} \approx 10.84$

$\therefore$ To minimize the cost of the metal, the radius should be 5.4 cm and the height should be 10.8 cm

6. Let $2x$ be the width of the rectangular portion of the window, and let y be the height of the rectangular portion.



To admit the greatest amount of light,

max *Area = Area of rectangle + Area of semi-circle*

$$A = 2xy + \frac{1}{2}\pi x^2 \qquad \text{(radius} = x\text{)}$$

given *Perimeter* $= 2x + 2y + \frac{1}{2}(2\pi x) = 8$

$$2y = 8 - 2x - \pi x$$

$$y = 4 - x - \tfrac{\pi}{2}x \quad \text{(sub into } A\text{)}$$

$$A = 2x\left(4 - x - \frac{\pi}{2}x\right) + \frac{1}{2}\pi x^2$$

$$= 8x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$$

$$= 8x - 2x^2 - \frac{1}{2}\pi x^2 \qquad \left(0 \le x \le \frac{8}{2+\pi} \approx 1.56\right)$$

$$A'(x) = 8 - 4x - \pi x$$

For critical numbers,

(i) $A'(x) = 8 - 4x - \pi x = 0$

$$x(4+\pi) = 8$$

$$x = \frac{8}{(4+\pi)}$$

$$x \approx 1.12 \text{ m}$$

(ii) $A'(x)$ is defined for all values of x in the domain

Check:

$$A(0) = 8(0) - 2(0)^2 - \tfrac{1}{2}\pi(0)^2 = 0 \text{ m}^2$$

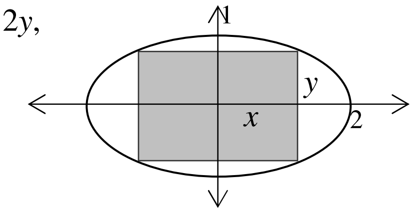
$$A(1.12) = 8(1.12) - 2(1.12)^2 - \tfrac{1}{2}\pi(1.12)^2 = 4.48 \text{ m}^2 \qquad \text{(max)}$$

$$A(1.56) = 8(1.56) - 2(1.56)^2 - \tfrac{1}{2}\pi(1.56)^2 = 3.79 \text{ m}^2$$

Determine width = $2x = 2(1.12) = 2.24$ m

$\therefore$ To admit the greatest amount of light, the width of the window should be 2.24 m.

7. Let the length of the rectangle be $2x$ and the width be $2y$, such that (x,y) is a point on the ellipse as shown.



max Area = $2x \cdot 2y = 4xy$

given $x^2 + 4y^2 = 4$

$$x^2 = 4 - 4y^2$$

$$x = \sqrt{4-4y^2} \quad \text{(sub into Area equation)}$$

Area = $A(y) = 4y\sqrt{4-4y^2} \qquad (0 \le y \le 1)$

$$A'(y) = 4\sqrt{4-4y^2} + 4y\left[\tfrac{1}{2}\left(4-4y^2\right)^{-\frac{1}{2}}(-8y)\right]$$

$$= 4\left(4-4y^2\right)^{-\frac{1}{2}}\left[\left(4-4y^2\right)-4y^2\right]$$

$$= 4\left(4-4y^2\right)^{-\frac{1}{2}}\left(4-8y^2\right)$$

$$= \frac{16\left(1-2y^2\right)}{\left(4-4y^2\right)^{\frac{1}{2}}}$$

For critical numbers,

(i) $A'(y) = \dfrac{16\left(1-2y^2\right)}{\left(4-4y^2\right)^{\frac{1}{2}}} = 0$

$$1 - 2y^2 = 0$$

$$y^2 = \frac{1}{2}$$

$$y = \frac{1}{\sqrt{2}}$$

(ii) $A'(y)$ is undefined when $y = 1$

Find dimensions:

$2x = 2\sqrt{4-4\left(\frac{1}{\sqrt{2}}\right)^2} = 2\sqrt{2}$

$2y = 2\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$

Check:

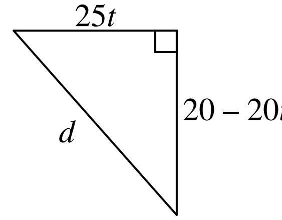
$$A(0) = 4(0)\sqrt{4-4(0)^2} = 0$$

$$A\left(\frac{1}{\sqrt{2}}\right) = 4\left(\frac{1}{\sqrt{2}}\right)\sqrt{4-4\left(\frac{1}{\sqrt{2}}\right)^2} = 4 \qquad \text{(max)}$$

$$A(1) = 4(1)\sqrt{4-4(1)^2} = 0$$

$\therefore$ The largest rectangle which can be inscribed in the given ellipse has an area of 4 units^2 when the dimensions of the rectangle is $2\sqrt{2}$ units $\times$ $\sqrt{2}$ units .

8. Let t be the time (in hours) since noon, when the boats are closest together.
Let d be the distance between them.



At 12:00 noon + t hours,
the westbound boat has travelled $25t$ km and
the northbound boat has travelled $20t$ km, so
it is $20 - 20t$ km from the dock.

Minimize d such that:
$$d^2 = (25t)^2 + (20 - 20t)^2$$
$$= 625t^2 + 400 - 800t + 400t^2$$
$$= 1025t^2 - 800t + 400 \qquad (0 \le t \le 1)$$

Find the derivative:
$$2d\frac{dd}{dt} = 2050t - 800$$
$$\frac{dd}{dt} = \frac{2050t - 800}{2d} = \frac{2050t - 800}{2\sqrt{(25t)^2 + (20 - 20t)^2}}$$

For critical numbers,

(i) $\dfrac{dd}{dt} = \dfrac{2050t - 800}{2\sqrt{(25t)^2 + (20 - 20t)^2}} = 0$
$$2050t - 800 = 0$$
$$t = \frac{800}{2050} \approx 0.39$$

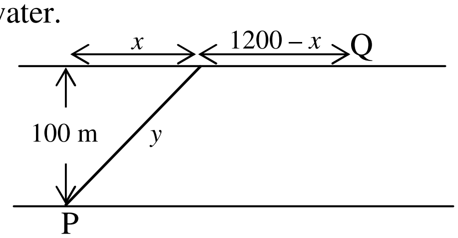
(ii) $\dfrac{dd}{dt}$ is defined for all values of t in the domain

Check:
$$d(0) = \sqrt{1025(0)^2 - 800(0) + 400} = 20 \text{ km}$$
$$d(0.39) = \sqrt{1025(0.39)^2 - 800(0.39) + 400} \approx 15.62 \text{ km} \qquad \text{(min.)}$$
$$d(1) = \sqrt{1025(1)^2 - 800(1) + 400} = 25 \text{ km}$$

$\therefore$ The boats are closest together at 0.39 hours after noon, ie. at approximately 12:23 pm.

Note: $0.39 \times 60 \approx 23$ minutes

9. Let (1200 – x) be the distance the cable should be run on land.
Let *y* be the distance the cable should be run under water.



From the diagram,

$$y^2 = x^2 + 10000$$
$$y = \sqrt{x^2 + 10000}$$

Minimize cost such that

$$C = 40(1200 - x) + 80y$$
$$= 48000 - 40x + 80\sqrt{x^2 + 10000} \qquad (0 \le x \le 1200)$$

Find the derivative: $C'(x) = -40 + 40\left(x^2 + 10000\right)^{-\frac{1}{2}}(2x)$

$$= -40 + \frac{80x}{\sqrt{x^2 + 10000}}$$

For critical numbers,

(i) $C'(x) = -40 + \dfrac{80x}{\sqrt{x^2 + 10000}} = 0$ (ii) $C'(x)$ is defined for all x values in domain

$$\frac{80x}{\sqrt{x^2 + 10000}} = 40$$
$$80x = 40\sqrt{x^2 + 10000}$$
$$x = 0.5\sqrt{x^2 + 10000}$$
$$x^2 = 0.25\left(x^2 + 10000\right)$$
$$x^2 = 0.25x^2 + 2500$$
$$0.75x^2 = 2500$$
$$x^2 = \frac{10000}{3}$$
$$x = \frac{100}{\sqrt{3}} \approx 57.7$$

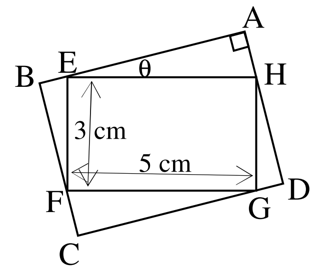
Check: $C(0) = 48000 - 40(0) + 80\sqrt{(0)^2 + 10000} = \$56000$

$C(57.7) = 48000 - 40(57.7) + 80\sqrt{(57.7)^2 + 10000} \approx \$54928.20$ (min)

$C(1200) = 48000 - 40(1200) + 80\sqrt{(1200)^2 + 10000} = \$96332.76$

$\therefore$ To minimize the cost of laying the cable, it should be installed from Q to a point 1142 m east, then 115.5 m under water to point P.

10. a) Define points E, F, G, and H as shown in diagram.



Note: $\theta = \angle AEH = \angle CGF \qquad \left(0 \le \theta \le \frac{\pi}{2}\right)$

$$\text{Area of } ABCD = |AB| \times |BC| = \left(|AE| + |EB|\right) \times \left(|BF| + |FC|\right)$$

From ΔAEH,

$$\frac{|AE|}{5} = \cos\theta$$
$$|AE| = 5\cos\theta$$

From ΔFCG,

$$\frac{|FC|}{5} = \sin\theta$$
$$|FC| = 5\sin\theta$$

From ΔEBF, $\angle BEF = \frac{\pi}{2} - \theta$

$$\frac{|EB|}{3} = \cos\left(\frac{\pi}{2} - \theta\right)$$
$$\frac{|EB|}{3} = \sin\theta$$
$$|EB| = 3\sin\theta$$

$$\frac{|BF|}{3} = \sin\left(\frac{\pi}{2} - \theta\right)$$
$$\frac{|BF|}{3} = \cos\theta$$
$$|BF| = 3\cos\theta$$

This step uses Complementary Identities (see pg. 391 of text)

From above,
$$\begin{aligned}\text{Area of } ABCD &= \left(|AE| + |EB|\right) \times \left(|BF| + |FC|\right) \\ &= \left(5\cos\theta + 3\sin\theta\right)\left(3\cos\theta + 5\sin\theta\right) \\ &= 15\cos^2\theta + 25\sin\theta\cos\theta + 9\sin\theta\cos\theta + 15\sin^2\theta \\ &= 15\left(\sin^2\theta + \cos^2\theta\right) + 34\sin\theta\cos\theta \\ &= 15(1) + 17(2\sin\theta\cos\theta) \\ A(\theta) &= 15 + 17\sin 2\theta\end{aligned}$$

$$\frac{dA}{d\theta} = 34\cos 2\theta$$

For critical numbers,

(i) $\frac{dA}{d\theta} = 34\cos 2\theta = 0$

$$\cos 2\theta = 0$$
$$2\theta = \frac{\pi}{2}$$
$$\theta = \frac{\pi}{4}\text{ rad}$$

(ii) $\frac{dA}{d\theta}$ is defined for all x values in domain

Check: $A(0) = 15 + 17\sin[2(0)] = 15\text{ cm}^2$

$A\left(\frac{\pi}{4}\right) = 15 + 17\sin\left[2\left(\frac{\pi}{4}\right)\right] = 32\text{ cm}^2$ (max)

$A\left(\frac{\pi}{2}\right) = 15 + 17\sin\left[2\left(\frac{\pi}{2}\right)\right] = 15\text{ cm}^2$

$\therefore$ The maximum area is 32 cm^2 when $\theta = \dfrac{\pi}{4}$ radians.

b) From part a), in rectangle ABCD,

length = $|AE| + |EB|$
$= 5\cos\theta + 3\sin\theta$

width = $|BF| + |FC|$
$= 3\cos\theta + 5\sin\theta$

When $\theta = \frac{\pi}{4}$

length $= 5\cos\left(\frac{\pi}{4}\right) + 3\sin\left(\frac{\pi}{4}\right)$
$= 5\left(\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right)$
$= \dfrac{8}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = 4\sqrt{2}$

width $= 3\cos\left(\frac{\pi}{4}\right) + 5\sin\left(\frac{\pi}{4}\right)$
$= 3\left(\frac{1}{\sqrt{2}}\right) + 5\left(\frac{1}{\sqrt{2}}\right)$
$= \dfrac{8}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = 4\sqrt{2}$

$\therefore$ For maximum area, the dimensions of rectangle ABCD should be $4\sqrt{2}$ cm $\times$ $4\sqrt{2}$ cm.